INTRODUCTION

Let S_n be the symmetric group on n letters, X be a space, and Sp^nX be the orbit space X^n/S_n of the S_n action on X^n by permuting the coordinates. If $X = \{X_k, \varepsilon_k\}$ is a spectrum with structure maps $\varepsilon_k : \Sigma X_k \to X_{k+1}$, then Sp^nX is the spectrum $\{Sp^nX_k, Sp^n(\varepsilon_k) \circ f_k\}$, where $f_k : \Sigma Sp^nX_k \to Sp^n\Sigma X_k$ is defined by $f_k(t\wedge(x_1, \ldots, x_n)) = (t\wedge x_1, \ldots, t\wedge x_n)$. Let Sp^nS^0 denote the n-fold symmetric product of the sphere spectrum. There is a natural inclusion $Sp^nS^0 \hookrightarrow Sp^{n+1}S^0$ by sending extra coordinate to base point. Let $D(k)$ be the cofiber of the diagonal map $d : Sp^{2^{k-1}}S^0 \to Sp^{2^k}S^0$. Let $M(n) = \Sigma^{-n}(D(n)/D(n-1))$. Let $L(k) = \Sigma^{-k}(Sp^{2^k}S^0/Sp^{2^{k-1}}S^0)$. Mitchell and Priddy [1] showed that $M(n) \cong L(n) \vee L(n-1)$. For further details the reader may consult [1]. Thus we study on $L(n)$. James, Thomas, Toda and Whitehead [3] showed that $\Sigma^{-1}Sp^2S^0/S^0 \simeq RP^\infty$. Here we show that the connective K-theory of $L(k)$ splits into copies of $HZ/2$ for $k = 2, 3$.

Here and throughout, all spaces are localized at prime 2, $\widetilde{H}^*(X)$ means the reduced mod 2 cohomology of X, $Z/2$ is the cyclic group of order 2, A is the mod 2 Steenrod algebra, and $E = E < Q_0, Q_1 >$ is the exterior algebra, which is a subalgebra of A, generated by $Q_0 = Sq^1$ and $Q_1 = Sq^3 + Sq^2Sq^1$. For the convenience, we denote $a \otimes b$ for $a \otimes_{Z/2} b$.

Johnson and Wilson [4] showed that $\widetilde{H}^*(RP^\infty \wedge RP^\infty) \cong M \oplus D$ as E-module, and $bu \wedge (RP^\infty \wedge RP^\infty) \simeq (\bigvee_\alpha \Sigma^{\dim\alpha}HZ/2) \vee (\Sigma^\beta bu \wedge RP^\infty)$, where each α is in the basis of the free E-module M, and β correspondes to the generators of D. Inductively we can see that $bu \wedge (\bigwedge^n RP^\infty) \simeq (\bigvee_\alpha \Sigma^\alpha HZ/2) \vee (\Sigma^\beta bu \wedge RP^\infty)$ for some appropriate families of α and β corresponding respectively to the generators of each splitting of $\widetilde{H}^*(\bigwedge^n RP^\infty)$ as E-module. Similarly Yan [6] showed that $bu \wedge BO(n) \simeq (\bigvee_\alpha \Sigma^\alpha HZ/2) \vee (\bigvee_\beta \Sigma^\beta bu) \vee (\bigvee_\gamma \Sigma^\gamma bu \wedge RP^\infty)$ for some appropriate families of α, β, and γ corresponding respectively to the generators of each splitting of $\widetilde{H}^*(BO(n))$ as E-module. Moreover, Mitchell and Priddy [1] showed that $M(n)$ is a stable summand of both $BO(n)$ and $\bigwedge^n RP^\infty$, and so is $L(n)$. Thus $bu \wedge L(n)$ should split and this splitting might depend on the splitting of $\widetilde{H}^*(L(n))$.

Let $X = RP^\infty \wedge RP^\infty$. Recall Johnson and Wilson's work [4] to split $bu \wedge X$. First split $\widetilde{H}^*(X)$ into D and a free part M as E-module. Secondly construct a space Y such that $\widetilde{H}^*(bu \wedge Y) \cong (A \otimes_E Z/2) \otimes D$. The third step is to determine α such that $\widetilde{H}^*(\bigvee_\alpha \Sigma^\alpha HZ/2) \cong (A \otimes_E Z/2) \otimes M \cong A \otimes_E M$. The last isomorphism is followed by the Proposition 1.7 of Liulevicius [5]. Finally construct the homotopy equivalence of $bu \wedge X$ and $(\bigvee_\alpha \Sigma^\alpha HZ/2) \vee (bu \wedge Y)$.

Now we just let $X = L(k)$ for $k = 2, 3$, and follow the steps above. First we show

Theorem 1. $\widetilde{H}^*(L(2))$ is a free E-module.

Then we have

Corollary 1. $bu \wedge L(2) \simeq \vee_\alpha \Sigma^\alpha HZ/2$ where $\alpha = 2i + 4j + 2$ for $2i > 2j + 1 > 0$.

Let θ be the admissable element of A with length k and x_i is the generator of $\widetilde{H}^1(\overset{k}{\times} RP^\infty)$ for $i = 1 \cdots k$. Mitchell and Priddy [1] showed that $M(k)$ is a stable summand of $\overset{k}{\times} RP^\infty$, and the $Z/2$-basis of $\widetilde{H}^*(M(k))$ embedded in $\widetilde{H}^*(\overset{k}{\times} RP^\infty)$ is $\{\theta(x_1^{-1} \cdots x_k^{-1})\}$. Since $L(k)$ is a summand of $M(k)$, and generators of $\widetilde{H}^*(L(k))$ are $\{\theta \in A - ASq^1\}$, the $Z/2$-basis of $\widetilde{H}^*(L(k))$ embedded in $\widetilde{H}^*(\overset{k}{\times} RP^\infty)$ is $\{\theta(x_1^{-1} \cdots x_k^{-1}) \mid \theta \in A - ASq^1\}$. We compute the generator $\theta(x_1^{-1} \cdots x_k^{-1})$ for $\theta \in A - ASq^1$ here. Let $c_1 > 0$, $c_i > \sum_{j<i} c_j + 1$, $I = (\Sigma_{i=1}^k c_i + 1, \Sigma_{i=1}^{k-1} c_i + 1, \cdots, \Sigma_{i=1}^2 c_i + 1, c_1 + 1)$, and $Sq^I = Sq^{\Sigma_{i=1}^k c_i + 1} \cdots Sq^{c_1+1}$. Note that $Sq^j x^{-1} = x^{j-1}$ for $\dim x = 1$. Then

Theorem 2. $Sq^I(x_1^{-1} \cdots x_k^{-1}) = \sum_{\sigma \in S_k} \Pi_{i=1}^k X_i^{\gamma(i)}$, where $X_i = h(i,i)$, $h(l,t) = h(l-1,t)(h(l-1,l-1) + h(l-1,t))$, $h(1,t) = x_{\sigma(t)}$, and $\gamma(i) = \Sigma_{j=1}^{k-i+1} c_j - \Sigma_{j=1}^{k-i} \Sigma_{k=1}^j c_k$.

By the theorem 2, we have the generator of $\widetilde{H}^*(L(3))$ embedded in $\widetilde{H}^*(\overset{3}{\times} RP^\infty)$. Then we can show

Theorem 3. $\widetilde{H}^*(L(3))$ is a free E-module.

With the analogous argument in the Corollary 1, we have

Corollary 3.1 $bu \wedge L(3) \simeq (\vee_\alpha \Sigma^\alpha HZ/2\ ) \vee (\vee_\beta \Sigma^\beta HZ/2)$ where $\alpha = 2i+4j+6k+2$ for $2i > 2j+2k+1$ and $2j+1 > 2k > 0$, and $\beta = 2i'+4j'+6k'+3$ for $2i' > 2j'+2k'+1$ and $2j' > 2k'+1 > 0$.

Since $M(3) \cong L(3) \vee L(2)$, we have

Corollary 3.2 $M(3) \simeq (\vee_\alpha \Sigma^\alpha HZ/2\ ) \vee (\vee_\beta \Sigma^\beta HZ/2) \vee (\vee_\gamma \Sigma^\gamma HZ/2)$ where $\alpha = 2i + 4j + 6k + 2$ for $2i > 2j + 2k + 1$ and $2j + 1 > 2k > 0$, $\beta = 2i'+4j'+6k'+3$ for $2i' > 2j'+2k'+1$ and $2j' > 2k'+1 > 0$, and $\gamma = 2i''+4j''+2$ for $2i'' > 2j''+1 > 0$.