2. PROOF OF THEOREM 2

Recall that the $Z/2$-basis of $\widetilde{H}^*(L(k))$ embedded in $\widetilde{H}^*(\overset{k}{\times} RP^{\infty})$ is $\{\theta(x_1^{-1} \cdots x_k^{-1}) \mid \theta \in A - ASq^1\}$. We are going to compute $\theta(x_1^{-1} \cdots x_k^{-1})$ for $\theta \in A - ASq^1$. Let $c_1 > 0$, $c_i > \sum\limits_{j<i} c_j + 1$, $\alpha_t = \Sigma_{i=1}^t c_i + 1$, $I = (\alpha_k, \alpha_{k-1}, \cdots, \alpha_1)$, and $Sq^I = Sq^{\Sigma_{i=1}^k c_i + 1} \cdots Sq^{c_1+1}$. For $1 \le s \le k$, let $I^{(s)} = (\alpha_k, \alpha_{k-1}, \cdots, \alpha_s)$. Denote the length of I by $l(I)$. Note that I and $I^{(s)}$ are admissible. Hence Sq^I may represent $\theta \in A - ASq^1$ with length k.

Lemma 2.1 Let $\gamma_i \geq -1$, for $i = 1 \cdots k$, and at least one γ_i is nonzero. Let $l(I) \leq k$, and $\Lambda = \{\gamma_i = -1\}$. If $|\Lambda| < l(I)$ and $\Sigma_{i=1}^k \gamma_i < \alpha_1$, $Sq^I(x_1^{\gamma_1} \cdots x_k^{\gamma_k}) = 0$.

Proof. Use induction on $l(I)$. For $l(I) = 1$, use induction on k. For $k = 1$, $\gamma_1 < \alpha_1$, $Sq^{\alpha_1}(x_1^{\gamma_1}) = 0$. For $k > 1$, assume the hypothesis true for $k - 1$. Without loss of generality, we assume $\gamma_k > 0$.

$Sq^{\alpha_1}(x_1^{\gamma_1} \cdots x_k^{\gamma_k})$
$= \Sigma_{j=0}^{\alpha_1} Sq^j(x_k^{\gamma_k}) Sq^{\alpha_1 - j}(x_1^{\gamma_1} \cdots x_{k-1}^{\gamma_{k-1}})$
$= \Sigma_{j=0}^{\gamma_k} \binom{\gamma_k}{j} x_k^{\gamma_k + j} Sq^{\alpha_1 - j}(x_1^{\gamma_1} \cdots x_{k-1}^{\gamma_{k-1}})$.

Since $\Sigma_{i=1}^k \gamma_i < \alpha_1$, $\Sigma_{i=1}^{k-1} \gamma_i < \alpha_1 - \gamma_k \leq \alpha_1 - j$. Thus $Sq^{\alpha_1 - j}(x_1^{\gamma_1} \cdots x_{k-1}^{\gamma_{k-1}}) = 0$ for $0 \le j \le \gamma_k$. Hence $Sq^{\alpha_1}(x_1^{\gamma_1} \cdots x_k^{\gamma_k}) = 0$.

Assume the hypothesis true for $l(I) = t - 1 < k$. Let $l(I) = t \leq k$.

$Sq^I(x_1^{\gamma_1} \cdots x_k^{\gamma_k})$
$= Sq^{I^{(2)}} Sq^{\alpha_1}(x_1^{\gamma_1} \cdots x_k^{\gamma_k})$
$= \sum\limits_{\{\gamma_1', \cdots \gamma_k'\}} Sq^{I^{(2)}}(x_1^{\gamma_1'} \cdots x_k^{\gamma_k'})$.

For each collection $\{\gamma_1', \cdots \gamma_k'\}$, let $\Lambda' = \{\gamma_i' = -1\}$. $|\Lambda'| < |\Lambda| \leq t - 1$, and $\Sigma_{i=1}^k \gamma_i' = \Sigma_{i=1}^k \gamma_i + \alpha_1 < 2\alpha_1 < \alpha_2$, thus $Sq^{I^{(2)}}(x_1^{\gamma_1'} \cdots x_k^{\gamma_k'}) = 0$ for each collection $\{\gamma_1', \cdots \gamma_k'\}$. Hence $Sq^{I^{(2)}}(x_1^{\gamma_1} \cdots x_k^{\gamma_k}) = 0$. ■

This tells us that to compute $Sq^I(x_1^{-1} \cdots x_k^{-1})$ it suffices to consider those special terms which have enough number of terms of value -1 that appear in the power. According to the Cartan formula, these special terms are symmetric. For example, let $k = 2$.

$Sq^i Sq^j(x_1^{-1} x_2^{-1})$
$= Sq^i(\Sigma_{l=0}^j Sq^{j-l}(x_1^{-1}) Sq^l(x_2^{-1}))$
$= Sq^i(\Sigma_{l=0}^j x_1^{j-l-1} x_2^{l-1})$
$= Sq^i(x_1^{j-1} x_2^{-1} + x_1^{-1} x_2^{j-1} + \Sigma_{l=1}^{j-1} x_1^{j-l-1} x_2^{l-1})$.

Here $Sq^i(\Sigma_{l=1}^{j-1}x_1^{j-l-1}x_2^{l-1}) = 0$, hence it only need to consider $Sq^i(x_1^{j-1}x_2^{-1} + x_1^{-1}x_2^{j-1})$ where $x_1^{j-1}x_2^{-1}$ and $x_1^{-1}x_2^{j-1}$ are symmetric.

Theorem 2. $Sq^I(x_1^{-1}\cdots x_k^{-1}) = \sum\limits_{\sigma\in S_k}\Pi_{i=1}^k X_i^{\gamma(i)}$, where $X_i = h(i,i)$, $h(l,t) = h(l-1,t)(h(l-1,l-1)+h(l-1,t))$, $h(1,t) = x_{\sigma(t)}$, and $\gamma(i) = \Sigma_{j=1}^{k-i+1}c_j - \Sigma_{j=1}^{k-i}\Sigma_{k=1}^{j}c_k$.

Proof. For $i = 1\cdots k-1$, let Λ_i be the cyclic subgroup of S_k generated by $(1,2,\cdots,k-i+1)\}$. Let $\Gamma_i = \{\sigma\in S_k \mid \sigma = \nu_1\cdots\nu_i$, where $\nu_j\in\Lambda_j$ for each $j\}$.

Let $f(\alpha,\beta) = \Sigma_{i=1}^{\beta}c_i + \Sigma_{i=\beta+1}^{\alpha-1}j_{i,\beta} - \Sigma_{i=1}^{\beta-1}j_{\beta,i}$, and $g(\gamma) = \Sigma_{i=1}^{k-\gamma+1}c_i + \Sigma_{i=k-\gamma+2}^{k}j_{i,k-\gamma+1} - \Sigma_{i=1}^{k-\gamma}j_{k-\gamma+1,i}$. By the Cartan formula, we have

$Sq^I(x_1^{-1}\cdots x_k^{-1})$
$= Sq^{I^{(2)}}Sq^{\alpha_1}(x_1^{-1}\cdots x_k^{-1})$
$= Sq^{I^{(2)}}\Sigma_{j=0}^{\alpha_1}Sq^{\alpha_1-j}(x_1^{-1}\cdots x_{k-1}^{-1})Sq^j(x_k^{-1})$
$= Sq^{I^{(2)}}\Sigma_{j=0}^{\alpha_1}Sq^{\alpha_1-j}(x_1^{-1}\cdots x_{k-1}^{-1})x_k^{j-1}$
$= Sq^{I^{(2)}}\sum\limits_{\sigma\in\Lambda_1}x_{\sigma(1)}^{-1}\cdots x_{\sigma(k-1)}^{-1}x_{\sigma(k)}^{c_1}$ (by lemma 2.1)
$= \sum\limits_{\sigma_1\in\Lambda_1}Sq^{I^{(3)}}\Sigma_{j_{2,1}=0}^{c_1}\binom{c_1}{j_{2,1}}x_{\sigma_1(k)}^{c_1+j_{2,1}}Sq^{\alpha_2-j_{2,1}}(x_{\sigma_1(1)}^{-1}\cdots x_{\sigma_1(k-1)}^{-1})$
$= \sum\limits_{\sigma_1\in\Lambda_1}\Sigma_{j_{2,1}=0}^{c_1}\binom{c_1}{j_{2,1}}Sq^{I^{(3)}}(\sum\limits_{\sigma\in\Lambda_2}x_{\sigma_1\sigma(1)}^{-1}\cdots x_{\sigma_1\sigma(k-1)}^{\Sigma_{i=1}^2c_i-j_2}x_{\sigma_1\sigma(k)}^{c_1+j_{2,1}})$
(by lemma 2.1)
$= \sum\limits_{\sigma\in\Gamma_2}\Sigma_{j_{2,1}=0}^{c_1}\binom{c_1}{j_{2,1}}Sq^{I^{(3)}}(x_{\sigma(1)}^{-1}\cdots x_{\sigma(k-1)}^{\Sigma_{i=1}^2c_i-j_{2,1}}x_{\sigma(k)}^{c_1+j_{2,1}})$
$= \cdots$
$= \sum\limits_{\sigma\in\Gamma_{t-1}}$
$\Sigma_{j_{2,1}=0}^{c_1}\Sigma_{j_{3,1}=0}^{c_1+j_2}\Sigma_{j_{3,2}=0}^{\Sigma_{i=1}^2c_i-j_2}\cdots\Sigma_{j_{t-1,1}=0}^{f(t-1,1)}\Sigma_{j_{t-1,2}=0}^{f(t-1,2)}\cdots\Sigma_{j_{t-1,t-2}=0}^{f(t-1,t-2)}$
$\Pi_{\alpha=2}^{t-1}\Pi_{\beta=1}^{\alpha}\binom{f(\alpha,\beta)}{j_{\alpha,\beta}}$
$Sq^{I^{(t)}}(x_{\sigma(1)}^{-1}\cdots x_{\sigma(k-t+1)}^{-1}$
$x_{\sigma(k-t+2)}^{\Sigma_{i=1}^{t-1}c_i-\Sigma_{i=1}^{t-2}j_{t-1,i}}\cdots x_{\sigma(k-s)}^{\Sigma_{i=1}^{s+1}c_i-\Sigma_{i=1}^{s}j_{s+1,i}+\Sigma_{i=s+2}^{t-1}j_{i,s+1}}\cdots x_{\sigma(k)}^{c_1+\Sigma_{i=2}^{t-1}j_{i,1}})$
$= \cdots$
$= \sum\limits_{\sigma\in\Gamma_k}$
$\Sigma_{j_{2,1}=0}^{f(2,1)}\Sigma_{j_{3,1}=0}^{f(3,1)}\Sigma_{j_{3,2}=0}^{f(3,2)}\cdots$
$\Sigma_{j_{t-1,1}=0}^{f(t-1,1)}\Sigma_{j_{t-1,2}=0}^{f(t-1,2)}\cdots\Sigma_{j_{t-1,t-2}=0}^{f(t-1,t-2)}\cdots$
$\Sigma_{j_{k,1}=0}^{f(k,1)}\Sigma_{j_{k,2}=0}^{f(k,2)}\cdots\Sigma_{j_{k,k-1}=0}^{f(k,k-1)}\Pi_{\alpha=2}^{k}\Pi_{\beta=1}^{\alpha}\binom{f(\alpha,\beta)}{j_{\alpha,\beta}}\Pi_{\gamma=1}^{k}x_{\sigma(\gamma)}^{g(\gamma)}$

Note that σ is exactly the permutation of k different numbers, thus we have

$$
\begin{aligned}
&\sum_{\sigma\in S_k} \\
&\Sigma_{j_{2,1}=0}^{f(2,1)}\Sigma_{j_{3,1}=0}^{f(3,1)}\Sigma_{j_{3,2}=0}^{f(3,2)}\cdots \\
&\Sigma_{j_{t-1,1}=0}^{f(t-1,1)}\Sigma_{j_{t-1,2}=0}^{f(t-1,2)}\cdots\Sigma_{j_{t-1,t-2}=0}^{f(t-1,t-2)}\cdots \\
&\Sigma_{j_{k,1}=0}^{f(k,1)}\Sigma_{j_{k,2}=0}^{f(k,2)}\cdots\Sigma_{j_{k,k-1}=0}^{f(k,k-1)}\Pi_{\alpha=2}^{k}\Pi_{\beta=1}^{\alpha}\binom{f(\alpha,\beta)}{j_{\alpha,\beta}}\Pi_{\gamma=1}^{k}x_{\sigma(\gamma)}^{g(\gamma)} \\
&= \sum_{\sigma\in S_k} \\
&\Sigma_{j_{2,1}=0}^{f(2,1)}\Sigma_{j_{3,1}=0}^{f(3,1)}\Sigma_{j_{3,2}=0}^{f(3,2)}\cdots \\
&\Sigma_{j_{t-1,1}=0}^{f(t-1,1)}\Sigma_{j_{t-1,2}=0}^{f(t-1,2)}\cdots\Sigma_{j_{t-1,t-2}=0}^{f(t-1,t-2)}\cdots \\
&\Sigma_{j_{k-1,1}=0}^{f(k-1,1)}\Sigma_{j_{k-1,2}=0}^{f(k-1,2)}\cdots\Sigma_{j_{k-1,k-2}=0}^{f(k-1,k-2)}\Pi_{\alpha=2}^{k-1}\Pi_{\beta=1}^{\alpha}\binom{f(\alpha,\beta)}{j_{\alpha,\beta}} \\
&x_{\sigma(1)}^{\Sigma_{j=1}^{k}c_j-\Sigma_{j=1}^{k-1}\Sigma_{k=1}^{j}c_k}\Pi_{i=2}^{k}(x_{\sigma(i)}(x_{\sigma(1)}+x_{\sigma(i)}))^{f(k,k-i+1)} \\
&=\cdots \\
&= \sum_{\sigma\in S_k} \\
&\Sigma_{j_{2,1}=0}^{f(2,1)}\Sigma_{j_{3,1}=0}^{f(3,1)}\Sigma_{j_{3,2}=0}^{f(3,2)}\cdots \\
&\Sigma_{j_{k-t,1}=0}^{f(k-t,1)}\Sigma_{j_{k-t,2}=0}^{f(k-t,2)}\cdots\Sigma_{j_{k-t,k-t-1}=0}^{f(k-t,k-t-1)}\Pi_{\alpha=2}^{k-t}\Pi_{\beta=1}^{\alpha}\binom{f(\alpha,\beta)}{j_{\alpha,\beta}} \\
&\Pi_{i=1}^{t}X_{\sigma(i)}^{\gamma(i)} \\
&\Pi_{j=t+1}^{k}(X_{\sigma(j)}(h(i,j)+X_{\sigma(j)}))^{f(k-t+1,k-j+1)} \\
&=\cdots \\
&= \sum_{\sigma\in S_k}\Pi_{i=1}^{k}X_i^{\gamma(i)}. \qquad \blacksquare
\end{aligned}
$$

For $k=3$, we immediately have

Corollary 2. The generator of $\widetilde{H}^*(L(3))$ embedded in $\widetilde{H}^*(\overset{3}{\times}RP^{\infty})$ is $\sum_{\sigma\in S_3} x_{\sigma(1)}^{c_1-c_3}(x_{\sigma(2)}\ (x_{\sigma(1)}+x_{\sigma(2)}))^{c_2}(x_{\sigma(3)}^2(x_{\sigma(1)}+x_{\sigma(3)})^2+x_{\sigma(2)}x_{\sigma(3)}(x_{\sigma(1)}+x_{\sigma(2)})(x_{\sigma(1)}+x_{\sigma(3)}))^{c_3}$, for $c_1>c_2+c_3$ and $c_2>c_3>0$.