3. PROOF OF THEOREM 3

Denote $a_{(c_1,c_2,c_3)} = \sum_{\sigma\in S_3} x_{\sigma(1)}^{c_1-c_3}(x_{\sigma(2)}(x_{\sigma(1)} + x_{\sigma(2)}))^{c_2}(x^2_{\sigma(3)}(x_{\sigma(1)} + x_{\sigma(3)})^2$ $+x_{\sigma(2)}\, x_{\sigma(3)}(x_{\sigma(1)}+x_{\sigma(2)})(x_{\sigma(1)}+x_{\sigma(3)}))^{c_3}$. Before showing $\widetilde{H}^*(L(3))$ is free E-module, for the conveniency, we compute $(x^2_{\sigma(3)}(x_{\sigma(1)}+x_{\sigma(3)})^2+x_{\sigma(2)}x_{\sigma(3)}(x_{\sigma(1)}+x_{\sigma(2)})(x_{\sigma(1)} + x_{\sigma(3)}))$ first.

Lemma 3.1 $Q_0(x^2_{\sigma(3)}(x_{\sigma(1)} + x_{\sigma(3)})^2 + x_{\sigma(2)}x_{\sigma(3)}(x_{\sigma(1)} + x_{\sigma(2)})(x_{\sigma(1)} + x_{\sigma(3)})) = 0$, $Q_1(x^2_{\sigma(3)}(x_{\sigma(1)} + x_{\sigma(3)})^2 + x_{\sigma(2)}x_{\sigma(3)}(x_{\sigma(1)} + x_{\sigma(2)})(x_{\sigma(1)} + x_{\sigma(3)})) = x_{\sigma(1)}x_{\sigma(2)}(x_{\sigma(1)} + x_{\sigma(2)})(x_{\sigma(2)}x_{\sigma(3)}(x_{\sigma(1)} + x_{\sigma(2)})(x_{\sigma(1)} + x_{\sigma(3)}) + x^2_{\sigma(3)}\ (x_{\sigma(1)} + x_{\sigma(3)})^2)$.

Proof. $Q_0(x^2_{\sigma(3)}(x_{\sigma(1)} + x_{\sigma(3)})^2 + x_{\sigma(2)}x_{\sigma(3)}(x_{\sigma(1)} + x_{\sigma(2)})(x_{\sigma(1)} + x_{\sigma(3)}))$
$= Q_0(x_{\sigma(2)}x_{\sigma(3)}(x_{\sigma(1)} + x_{\sigma(2)})(x_{\sigma(1)} + x_{\sigma(3)}))$
$= x_{\sigma(2)}x_{\sigma(3)}(x_{\sigma(1)} + x_{\sigma(2)})(x_{\sigma(1)} + x_{\sigma(3)})(x_{\sigma(2)} + x_{\sigma(3)} + (x_{\sigma(1)} + x_{\sigma(2)}) + (x_{\sigma(1)} + x_{\sigma(3)}))$
$= 0$.
$Q_1(x^2_{\sigma(3)}(x_{\sigma(1)} + x_{\sigma(3)})^2 + x_{\sigma(2)}x_{\sigma(3)}(x_{\sigma(1)} + x_{\sigma(2)})(x_{\sigma(1)} + x_{\sigma(3)}))$
$= Q_1(x_{\sigma(2)}x_{\sigma(3)}(x_{\sigma(1)} + x_{\sigma(2)})(x_{\sigma(1)} + x_{\sigma(3)}))$
$= x_{\sigma(2)}x_{\sigma(3)}(x_{\sigma(1)} + x_{\sigma(2)})(x_{\sigma(1)} + x_{\sigma(3)})(x^3_{\sigma(2)} + x^3_{\sigma(3)} + (x_{\sigma(1)} + x_{\sigma(2)})^3 + (x_{\sigma(1)} + x_{\sigma(3)})^3)$
$= x_{\sigma(2)}x_{\sigma(3)}(x_{\sigma(1)} + x_{\sigma(2)})(x_{\sigma(1)} + x_{\sigma(3)})(x_{\sigma(1)}x_{\sigma(2)}(x_{\sigma(1)} + x_{\sigma(2)}) + x_{\sigma(1)}\, x_{\sigma(3)}(x_{\sigma(1)} + x_{\sigma(3)}))$
$= x_{\sigma(1)}x_{\sigma(2)}(x_{\sigma(1)} + x_{\sigma(2)})(x_{\sigma(2)}x_{\sigma(3)}(x_{\sigma(1)} + x_{\sigma(2)})(x_{\sigma(1)} + x_{\sigma(3)}) + x^2_{\sigma(3)}\, (x_{\sigma(1)} + x_{\sigma(3)})^2)$. ■

As the case in $L(2)$, we can compute the E-module structure of $\widetilde{H}^*(L(3))$ directly. Here we give an example to show how easy it is to compute the action.

Example. $Q_1(a_{(2i,2j+1,2k)})$
$= \sum_{\sigma\in S_3} Q_1(x^{2i-2k}_{\sigma(1)}(x_{\sigma(2)}(x_{\sigma(1)} + x_{\sigma(2)}))^{2j+1}(x^2_{\sigma(3)}(x_{\sigma(1)} + x_{\sigma(3)})^2\ +x_{\sigma(2)}x_{\sigma(3)}\, (x_{\sigma(1)} + x_{\sigma(2)})(x_{\sigma(1)} + x_{\sigma(3)}))^{2k})$
$= \sum_{\sigma\in S_3} (x^{2i-2k}_{\sigma(1)} x^{2j}_{\sigma(2)}(x_{\sigma(1)}+x_{\sigma(2)})^{2j} Q_1(x_{\sigma(2)}(x_{\sigma(1)}+x_{\sigma(2)}))(x^2_{\sigma(3)}\ (x_{\sigma(1)}+x_{\sigma(3)})^2+ x_{\sigma(2)}x_{\sigma(3)}(x_{\sigma(1)} + x_{\sigma(2)})(x_{\sigma(1)} + x_{\sigma(3)}))^{2k}$
$= \sum_{\sigma\in S_3} (x^{2i-2k}_{\sigma(1)}(x_{\sigma(2)}(x_{\sigma(1)}+x_{\sigma(2)}))^{2j} x_{\sigma(2)}(x_{\sigma(1)}+x_{\sigma(2)})(x^3_{\sigma(2)} +(x_{\sigma(1)}+x_{\sigma(2)})^3)(x^2_{\sigma(3)}\, (x_{\sigma(1)} + x_{\sigma(3)})^2 + x_{\sigma(2)}x_{\sigma(3)}(x_{\sigma(1)} + x_{\sigma(2)})(x_{\sigma(1)} + x_{\sigma(3)}))^{2k})$

$= \sum_{\sigma\in S_3} (x_{\sigma(1)}^{2i-2k}(x_{\sigma(2)}(x_{\sigma(1)}+x_{\sigma(2)}))^{2j+1}(x_{\sigma(1)}^3+x_{\sigma(1)}x_{\sigma(2)}(x_{\sigma(1)}+x_{\sigma(2)}))(x_{\sigma(3)}^2$
$(x_{\sigma(1)}+x_{\sigma(3)})^2+x_{\sigma(2)}x_{\sigma(3)}(x_{\sigma(1)}+x_{\sigma(2)})(x_{\sigma(1)}+x_{\sigma(3)}))^{2k})$
$= a_{(2i+3,2j+1,2k)} + a_{(2i+1,2j+2,2k)}$.

Using these tools for computing, we have the following result.

Theorem 3. $\widetilde{H}^*(L(3))$ is the free E-module.

Proof. As in theorem 1, we show that the class $X = \{a_{(2i,2j+1,2k)}|2i > 2j+2k+1, 2j+1 > 2k > 0\} \cup \{a_{(2i,2j,2k+1)}|2i > 2j+2k+1, 2j > 2k+1 > 0\}$is an E-basis. It suffice to show each $a_{(\alpha,\beta,\gamma)}$ is generated uniquely. Since

$Q_0(a_{(2i,2j+1,2k)}) = a_{(2i+1,2j+1,2k)}$,
$Q_1(a_{(2i,2j+1,2k)}) = a_{(2i+3,2j+1,2k)} + a_{(2i+1,2j+2,2k)}$,
$Q_0Q_1(a_{(2i,2j+1,2k)}) = a_{(2i+2,2j+2,2k)}$,
$Q_0(a_{(2i,2j+1,2k)}) = a_{(2i+1,2j,2k+1)}$,
$Q_1(a_{(2i,2j+1,2k)}) = a_{(2i+3,2j,2k+1)} + a_{(2i+1,2j+1,2k+1)}$,
$Q_0Q_1(a_{(2i,2j,2k+1)}) = a_{(2i+2,2j+1,2k+1)}$, we see that each $a_{(\alpha,\beta,\gamma)}$ is generated uniquely. ■

Corollary 3.1 $bu \wedge L(3) \simeq (\vee_\alpha \Sigma^\alpha HZ/2\ ) \vee (\vee_\beta \Sigma^\beta HZ/2)$ where $\alpha = 2i+4j+6k+2$ for $2i > 2j+2k+1$ and $2j+1 > 2k > 0$, and $\beta = 2i'+4j'+6k'+3$ for $2i' > 2j'+2k'+1$ and $2j' > 2k'+1 > 0$.

Proof. Let a be the E-basis of $\widetilde{H}^*(L(3))$ and $g_a : L(3) \to \Sigma^{\dim(a)}HZ/2$ represent a. Construct the map g as follow:

$$\begin{aligned} g \quad : \quad & bu \wedge L(3) \stackrel{1_{bu}\wedge\vee_a g_a}{\longrightarrow} bu \wedge ((\vee_\alpha \Sigma^\alpha HZ/2) \vee (\vee_\beta \Sigma^\beta HZ/2)) \\ & \stackrel{\vee_a \nu}{\longrightarrow} (\vee_\alpha \Sigma^\alpha HZ/2) \vee (\vee_\beta \Sigma^\beta HZ/2) \end{aligned}$$

where $\nu : bu \wedge HZ/2 \to HZ/2$ is the map constructed as that in Corollary 1.

By the same argument as the proof of corollary 1, g is an equivalence. ■

Since $M(3) \cong L(3) \vee L(2)$, by the above corollary and the corollary 1, we have

Corollary 3.2 $M(3) \simeq (\vee_\alpha \Sigma^\alpha HZ/2\ ) \vee (\vee_\beta \Sigma^\beta HZ/2) \vee (\vee_\gamma \Sigma^\gamma HZ/2)$ where $\alpha = 2i + 4j + 6k + 2$ for $2i > 2j + 2k + 1$ and $2j + 1 > 2k > 0$, $\beta =$

$2i'+4j'+6k'+3$ for $2i' > 2j'+2k'+1$ and $2j' > 2k'+1 > 0$, and $\gamma = 2i''+4j''+2$ for $2i'' > 2j''+1 > 0$.