1. PROOF OF THEOREM 1

Mitchell and Priddy [2] showed that the $Z/2$-basis of $\widetilde{H}^*(L(2))$ embedded in $\widetilde{H}^*(RP^\infty \times RP^\infty)$ is $a_{(j,k)} = (x_1+x_2)^k(x_1^j x_2^k + x_1^k x_2^j)$ for $j > k > 0$, where x_1 and x_2 are the generators of $\widetilde{H}^1(RP^\infty \times RP^\infty)$. We observe the image of $\widetilde{H}^*(L(2))$ embedded in $\widetilde{H}^*(RP^\infty \times RP^\infty)$.

Since E is a subalgebra of A, $\widetilde{H}^*(X)$ is an E-module for any space or spectrum X. Recall that $Q_k(xy) = Q_k(x)y + xQ_k(x)$ for $k = 0, 1$ and $Q_0(x) = Sq^1 x = x^2$, $Q_1(x) = (Sq^3 + Sq^2Sq^1)(x) = x^4$ for $\dim x = 1$. With these rules, we are going to compute the E-module structure of $\widetilde{H}^*(L(2))$. First we need some observation.

Lemma 1.1 For $k = 0, 1$, $Q_k(\sum_i a_i \sum_j b_j) = (Q_k(\sum_i a_i))(\sum_j b_j) + (\sum_i a_i)(Q_k(\sum_j b_j))$.

Proof. $Q_k(\sum_{i,j} a_i b_j) = \sum_{i,j} Q_k(a_i b_j) = \sum_{i,j}(Q_k(a_i)b_j + a_i Q_k(b_j)) = \sum_{i,j} Q_k(a_i)b_j + \sum_{i,j} a_i Q_k(b_j) = (Q_k(\sum_i a_i))(\sum_j b_j) + (\sum_i a_i)(Q_k(\sum_j b_j))$. ■

With this lemma, if A and B are the combination of some monomials, $Q_k(\text{AB}) = Q_k(\text{A})\text{B} + \text{A}Q_k(\text{B})$. Thus we simplify $Q_k(\text{AB})$ to the action of Q_k on each factor of AB. Since $Q_k((x_1+x_2)^{2j}) = Q_k((x_1+x_2)^j)(x_1+x_2)^j + (x_1+x_2)^j Q_k((x_1+x_2)^j) = 0$ by the lemma above, and $Q_k((x_1+x_2)^{2j+1}) = (x_1+x_2)^{2j} Q_k((x_1+x_2))$ it suffices to consider the case $j = 0$. Therefor we consider $Q_k((x_1+x_2))$ next.

Lemma 1.2 $Q_0((x_1+x_2)) = (x_1+x_2)^2$, $Q_1((x_1+x_2)) = (x_1+x_2)^4$.

Proof. $Q_0((x_1+x_2)) = (x_1^2 + x_2^2) = (x_1+x_2)^2$.
$Q_1((x_1+x_2)) = (x_1^4 + x_2^4) = (x_1+x_2)^4$. ■

Let $a_{(j,k)} = (x_1+x_2)^k(x_1^j x_2^k + x_1^k x_2^j) = x_1^j x_2^k (x_1+x_2)^k + x_2^j x_1^k (x_1+x_2)^k = \sum_{\sigma\in S_2} x_{\sigma(1)}^j x_{\sigma(2)}^k (x_{\sigma(1)} + x_{\sigma(2)})^k$. By the above two lemmas, we can compute the E-module structure of $\widetilde{H}^*(L(2))$ directly by catching the parity of j and k, i.e. the power of each factor. For example,

$$\begin{aligned}
&Q_1(a_{(2i,2j+1)})\\
&= \sum_{\sigma\in S_2} Q_1(x_{\sigma(1)}^{2i} x_{\sigma(2)}^{2j+1}(x_{\sigma(1)} + x_{\sigma(2)})^{2j+1})\\
&= \sum_{\sigma\in S_2} x_{\sigma(1)}^{2i} x_{\sigma(2)}^{2j}(x_{\sigma(1)} + x_{\sigma(2)})^{2j} Q_1(x_{\sigma(2)}(x_{\sigma(1)} + x_{\sigma(2)}))
\end{aligned}$$

$= \sum\limits_{\sigma\in S_2} x^{2i}_{\sigma(1)}x^{2j}_{\sigma(2)}(x_{\sigma(1)}+x_{\sigma(2)})^{2j}(x^4_{\sigma(2)}(x_{\sigma(1)}+x_{\sigma(2)})+x_{\sigma(2)}(x_{\sigma(1)}+x_{\sigma(2)})^4)$
$= \sum\limits_{\sigma\in S_2} x^{2i}_{\sigma(1)}x^{2j+1}_{\sigma(2)}(x_{\sigma(1)}+x_{\sigma(2)})^{2j+1}(x^3_{\sigma(2)}+(x_{\sigma(1)}+x_{\sigma(2)})^3)$
$= \sum\limits_{\sigma\in S_2} x^{2i}_{\sigma(1)}x^{2j+1}_{\sigma(2)}(x_{\sigma(1)}+x_{\sigma(2)})^{2j+1}(x^3_{\sigma(1)}+x_{\sigma(1)}x_2(x_{\sigma(1)}+x_{\sigma(2)}))$
$= \sum\limits_{\sigma\in S_2} (x^{2i+3}_{\sigma(1)}x^{2j+1}_{\sigma(2)}(x_{\sigma(1)}+x_{\sigma(2)})^{2j+1}+x^{2i+1}_{\sigma(1)}x^{2j+2}_{2}(x_{\sigma(1)}+x_{\sigma(2)})^{2j+2})$
$= a_{(2i+3,2j+1)} + a_{(2i+1,2j+2)}$.
With these in mind, we can show the following theorem.

Theorem 1. $\widetilde{H}^*(L(2))$ is a free E-module.

Proof. We show that the class $X = \{a_{(2i,2j+1)} | 2i > 2j+1 > 0\}$ is an E-basis. Since $Q_0(a_{(2i,2j+1)}) = a_{(2i+1,2j+1)}$, $Q_1(a_{(2i,2j+1)}) = a_{(2i+3,2j+1)} + a_{(2i+1,2j+2)}$, $Q_0Q_1(a_{(2i,2j+1)}) = a_{(2i+2,2j+2)}$, we see that each $a_{(\alpha,\beta)}$ is generated uniquely. Hence $\widetilde{H}^*(L(2))$ is isomorphic to $\sum\limits_{\alpha\in X} E_\alpha$ for $E_\alpha = E$. Thus $\widetilde{H}^*(L(2))$ is a free E-module with basis X. ∎

Recall that $\widetilde{H}^*(bu) \cong A//A(Q_0,Q_1) \cong A\otimes_E Z/2$ where $A(Q_0,Q_1)$ is the ideal generated by Q_0 and Q_1. Let M and N be left A-modules with the actions μ_M and μ_N, then $M\otimes N$ is also a left A-module with the action defined by the composition

$$A\otimes(M\otimes N) \overset{\psi\otimes M\otimes N}{\rightarrow} A\otimes A\otimes(M\otimes N) \overset{A\otimes T\otimes N}{\rightarrow} A\otimes M\otimes A\otimes N \overset{A\otimes T\otimes N}{\rightarrow} M\otimes N,$$

where ψ is the diagonal map of A and $T(a\otimes b) = b\otimes a$ is the twist map. Denote $M\otimes N$ with this action by ${}_D(M\otimes N)$. Denote $M\otimes N$ with the extended A action over M by ${}_L(M\otimes N)$.

Let ν be the following composite map

$$\nu : bu\wedge HZ/2 \overset{i\wedge HZ/2}{\rightarrow} HZ/2\wedge HZ/2 \overset{\mu}{\rightarrow} HZ/2$$

where $i : bu \to HZ/2$ is the multiplicative map and μ is the ring structure map of $HZ/2$.

Corollary 1. $bu\wedge L(2) \simeq \bigvee\limits_{\alpha} \Sigma^\alpha HZ/2$ where $\alpha = 2i+4j+2$ for $2i > 2j+1 > 0$.

Proof. The proof is based on the concept of Yan's [6] for splitting $bu \wedge BO(2)$. Let $a \in \widetilde{H}^{2i+4j+2}(L(2))$ whose image in $\widetilde{H}^*(RP^\infty \times RP^\infty)$ is $a_{(2i,2j+1)}$, and let $g_a : L(2) \to \Sigma^{\dim(a)} HZ/2$ represent a. Construct the map g by the following composition:

$$g : bu \wedge L(2) \overset{1_{bu}\wedge \vee_a g_a}{\longrightarrow} bu \wedge (\bigvee_a \Sigma^{\dim(a)} HZ/2) \overset{\vee_a \nu}{\to} \vee_a \Sigma^{\dim(a)} HZ/2.$$

Note $\widetilde{H}^*(bu \wedge L(2)) \cong \widetilde{H}^*(bu) \otimes \widetilde{H}^*(L(2)) \cong (A \otimes_E Z/2) \otimes \widetilde{H}^*(L(2))$. Since ${}_D(\widetilde{H}^*(L(2)) \otimes (A \otimes_E Z/2)) \cong_D ((A \otimes_E Z/2) \otimes \widetilde{H}^*(L(2)))$ and ${}_D(\widetilde{H}^*(L(2)) \otimes Z/2) \cong \widetilde{H}^*(L(2))$, the isomorphism

$$\theta :_L (A \otimes_E \widetilde{H}^*(L(2))) \overset{\cong}{\to}_D ((A \otimes_E Z/2) \otimes \widetilde{H}^*(L(2)))$$

is given by $\theta(a \otimes x) = \Sigma a' \otimes 1 \otimes a'' x$, with the inverse $\theta^{-1}(a \otimes 1 \otimes x) = \Sigma a' \otimes \chi(a'') x$, where $\psi(a) = \Sigma a' \otimes a''$ and χ is the conjugation map(cf. [5],[6]). Hence $(A \otimes_E Z/2) \otimes \widetilde{H}^*(L(2)) \cong A \otimes_E \widetilde{H}^*(L(2))$.

Since $\widetilde{H}^*(L(2))$ is the free E-module by the theorem 1, $A \otimes_E \widetilde{H}^*(L(2))$ is a free A-module. $\widetilde{H}^*(bu \wedge L(2)) \cong (A \otimes_E Z/2) \otimes \widetilde{H}^*(L(2)) \cong A \otimes_E \widetilde{H}^*(L(2))$, thus $\widetilde{H}^*(bu \wedge L(2))$ is a free A-module. Consider

$$\begin{array}{ccccc} \widetilde{H}^*(\bigvee_a \Sigma^{\dim(a)} HZ/2) & \overset{g^*}{\to} & \widetilde{H}^*(bu \wedge L(2)) & \to & (A \otimes_E Z/2) \otimes \widetilde{H}^*(L(2)) \\ \Sigma^{\dim(a)} 1 & \to & 1 \otimes a & \to & 1 \otimes 1 \otimes a \end{array}$$

$$\begin{array}{cc} \overset{\theta^{-1}}{\to} & A \otimes_E \widetilde{H}^*(L(2)) \\ \to & 1 \otimes a \end{array}$$

where $\Sigma^{\dim(a)} 1$ is in the basis of $\widetilde{H}^*(\bigvee_a \Sigma^{\dim(a)} HZ/2)$. Thus $\widetilde{H}^*(\bigvee_a \Sigma^{\dim(a)} HZ/2)$ and $A \otimes_E \widetilde{H}^*(L(2))$ have the same rank. g induces an isomorphism in mod 2 cohomology and thus is an equivalence. ■